A Complex Version of G-Expectation and Its Application to Conformal Martingale

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Abstract

This paper is concerned with the connection between G-Brownian Motion and analytic functions. We introduce the complex version of sublinear expectation, and then do the stochastic analysis in this framework. Furthermore, the conformal G-Brownian Motion is introduced together with a representation, and the corresponding conformal invariance is shown.

Key words: G-expectation, complex sublinear expectation, G-Brownian motion, conformal martingale, conformal invariance

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1 Introduction

G-expectation was first introduced by peng to handle risk measures, stochastic volatility and model uncertainty(see[4],[5]). The idea of G-expectation lies in the new explanation of independence and distribution. The main difference between G-expectation and linear expectation is, of course, sublinearity, which is mathematically explained by a powerful tool in PDE, viscosity solution theory. A typical G-expectation is constructed on linear space of random variables $L_G^1(\Omega)$, the completion of $Lip(\Omega)$ of Lipschitz cylinder functions on d-dimensional continuous path space $\Omega = C_0^d[0, \infty)$. The corresponding G-Brownian Motion on this space is the canonical process, i.e. $B_t(\omega) = \omega(t), \omega \in \Omega$. A sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, Lip(\Omega))$ is defined as a viscosity solution of a heat equation, which is determined by a sublinear monotone function G. Then the random variable space $Lip(\Omega)$ is extended to completion by the norm $\hat{\mathbb{E}}[|\cdot|]$, and we denote the completion as $L_G^1(\Omega)$. By such construction, a time consistent conditional expectation is introduced naturally on $L_G^1(\Omega)$.

G-expectation theory is thriving in recent years.(see[2],[6],[7]) In classical case, we know there are many interesting connection between Brownian Motion and analytic functions. A fundamental one is Paul $L\acute{e}vy's$ theorem, that is, if

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$f(z)$ is analytic nonconstant function, and B_t is a two dimensional Brownian Motion, the process $f(B_t), t \geq 0$ is again Brownian Motion, probably moving at a variable speed.(see [1]) $L\acute{e}vy's$ result leads to the possibility to study analytic functions probabilistically. On the other hand, a beauty of linear expectation is that a distribution can be uniquely described by its characteristic function. However, there is little work on G-expectation under complex framework.

In this paper, we give the stochastic analysis of G-expectation in complex case, and furthermore, the conformal G-Brownian Motion and conformal martingales. We find that a special G-Brownian Motion, namely the conformal G-Brownian Motion still holds conformal invariance, i.e. it is still conformal after a transformation by a nonconstant analytic function. We should notice that conformal G-Brownian Motion is much more complicated than the classical case.

This paper is organized as follows. In section 2, we recall some basic results of G-Brownian. In section 3 and 4, we define the sublinear expectation under complex framework and the complex G-normal distribution and G-Brownian Motion. In the fifth part, we do the stochastic analysis in this framework and give It$\hat{o}'s$ formula. In the last part, we introduce the conformal martingale and prove conformal invariance.

2 Preliminaries

We start from some basic notes and results of G-expectation theory. More details can be read from [4], [5], [6].

2.1 Sublinear Expectation and G-Expectation

Let Ω be a given set and $\mathcal{H}$ be a linear space of real valued functions on Ω containing constants. Furthermore, suppose $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space Ω is viewed as sample space and $\mathcal{H}$ is the space of random variables.

Definition 2.1. *A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying*

(i) *constant preserving:*

$$\hat{\mathbb{E}}(c) = c, \qquad \forall\, c \in \mathbb{R}$$

(ii) *positive homogeneity:*

$$\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X), \qquad \lambda \geq 0 \quad X \in \mathcal{H}$$

(iii) *constant transferability:*

$$\hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}(X) + c, \qquad c \in \mathbb{R} \quad X \in \mathcal{H}$$

(iv) *monotonicity:*

$$\hat{\mathbb{E}}(X_1) \geq \hat{\mathbb{E}}(X_2) \qquad if\ X_1 \geq X_2$$

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2. *Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{b.Lip}(\mathbb{R}^n)$.*

Definition 2.3. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \cdot\cdot\cdot, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \cdot\cdot\cdot, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{b.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.*

Definition 2.4. *(maximal distribution) A d-dimensional random vector $\eta = (\eta_1, \cdots, \eta_d)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called maximal distributed if η satisfies:*

$$a\eta + b\bar{\eta} \overset{d}{=} (a+b)\eta,\ a, b \geq 0,$$

where $\bar{\eta}$ is an independent copy of η, i.e., $\bar{\eta} \overset{d}{=} \eta$ and $\bar{\eta} \perp \eta$.

Remark 2.5. *By the definition of maximal distribution, we can get that there exists a bounded, closed and convex subset $\Gamma \subset \mathbb{R}^d$ such that for any $\varphi \in C_{b.lip}(\mathbb{R}^d)$, we have:*

$$\hat{\mathbb{E}}[\varphi(\eta)] = \max_{y \in \Gamma} \varphi(y).$$

When $d = 1$, we have $\Gamma = [\underline{\mu}, \bar{\mu}]$, where $\bar{\mu} = \hat{\mathbb{E}}[\eta]$ and $\underline{\mu} = -\hat{\mathbb{E}}[-\eta]$. Furthermore, in this case we denote $\eta \overset{d}{=} N(\{\underline{\mu}, \bar{\mu}\} \times 0)$.

In the classical case, the maximal distributed random is a constant.

Definition 2.6. *(G-normal distribution) A d-dimensional random vector $X = (X_1, \cdot\cdot\cdot, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G-normally distributed if $\hat{\mathbb{E}}[|X|^3] < \infty$ and for each $a, b \geq 0$*

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X,$$

where $\bar{X}$ is an independent copy of X, and

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \to \mathbb{R},$$

Here $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

By[6]we know that $X = (X_1, \cdots, X_d)$ is G-normal distributed iff $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t} X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, is the viscosity solution of the following G-heat equation:

$$\partial_t u - G(D_x^2 u) = 0,\ u(0, x) = \varphi(x).$$

The function $G(\cdot) : \mathbb{S}_d \to \mathbb{R}$ is a monotonic, sublinear functional on $\mathbb{S}_d$, from which we can deduce that there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{B \in \Sigma} \operatorname{tr}[AB],$$

where $\mathbb{S}_d^+$ denotes the collection of nonnegative matrixes in $\mathbb{S}_d$.

Here is the typical construction of G-expectation from[5].For simplicity, we only consider the one dimensional condition.

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [(\max_{t \in [0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1].$$

and the canonical process is defined by $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$. Then we define

$$L_{ip}(\Omega) := \{\varphi(B_{t_1}, ..., B_{t_n}) : n \geq 1, t_1, ..., t_n \in [0, \infty), \varphi \in C_{b.Lip}(\mathbb{R}^n)\}.$$

Give a monotonic and sublinear function $G : \mathbb{S}_d \to \mathbb{R}$. The corresponding G-expectation $\hat{\hat{\mathbb{E}}}$ is a sublinear expectation on $L_{ip}(\Omega)$ satisfying:

$$\hat{\mathbb{E}}[X] = \tilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$ with $0 \leq t_0 < t_1 < \cdots < t_m < \infty$, where $\xi_1, \cdots, \xi_n$ are identically distributed d-dimensional G-normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \cdots, \xi_i)$ for every $i = 1, \cdots, m-1$.

Definition 2.7. *Let $\Omega_t = \{\omega_{\cdot \wedge t} : \omega \in \Omega\}$ for $t \geq 0$. For each $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, the conditional G-expectation of ξ under Ω_{t_i} is defined by*

$$\begin{aligned}
&\hat{\mathbb{E}}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})] \\
&= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),
\end{aligned}$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

For each fixed $T \geq 0$, we define

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) : n \geq 1, t_1, ..., t_n \in [0, T], \varphi \in C_{b.Lip}(\mathbb{R}^n)\}.$$

It is simple that $L_{ip}(\Omega_{T_1}) \subset L_{ip}(\Omega_{T_2}) \subset L_{ip}(\Omega)$ for $T_1 < T_2$. Furthermore, we denote $L_G^p(\Omega)$ and $L_G^p(\Omega_T)$, $p \geq 1$ as the completion of $L_{ip}(\Omega)$ and $L_{ip}(\Omega_T)$

under the norm $\|\xi\|_p = (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$. Consequently, $L_G^{p_1}(\Omega) \subset L_G^{p_2}(\Omega)$ for $p_1 \geq p_2 \geq 1$.

By the above construction we can get that the G-expectation $\hat{\mathbb{E}}[\cdot]$ can be continuously extended to a sublinear expectation on $(\Omega, L_G^1(\Omega))$ and it is still denoted by $\hat{\mathbb{E}}[\cdot]$. For each given $t \geq 0$, the conditional G-expectation $\hat{\mathbb{E}}_t[\cdot] : L_{ip}(\Omega) \to L_{ip}(\Omega_t)$ can be continuously extended as a mapping $\hat{\mathbb{E}}_t[\cdot] : L_G^1(\Omega) \to L_G^1(\Omega_t)$ and satisfies the following properties:

(i) If $X, Y \in L_G^1(\Omega)$, $X \geq Y$, then $\hat{\mathbb{E}}_t[X] \geq \hat{\mathbb{E}}_t[Y]$;

(ii) If $X \in L_G^1(\Omega_t)$, $Y \in L_G^1(\Omega)$, then $\hat{\mathbb{E}}_t[X + Y] = X + \hat{\mathbb{E}}_t[Y]$;

(iii) If $X, Y \in L_G^1(\Omega)$, then $\hat{\mathbb{E}}_t[X + Y] \leq \hat{\mathbb{E}}_t[X] + \hat{\mathbb{E}}_t[Y]$;

(iv) If $X \in L_G^1(\Omega_t)$ is bounded, $Y \in L_G^1(\Omega)$, then $\hat{\mathbb{E}}_t[XY] = X^+\hat{\mathbb{E}}_t[Y] + X^-\hat{\mathbb{E}}_t[-Y]$;

(v) If $X \in L_G^1(\Omega)$, then $\hat{\mathbb{E}}_s[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}_{s\wedge t}[X]$, in particular, $\hat{\mathbb{E}}[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}[X]$.

2.2 Stochastic calculus under G-Expectation

For a partition $\Pi = \{t_0, t_1, \cdots, t_N\}$ of $[0, T]$, we define

$$\eta_t(\omega) := \Sigma_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_k+1)}(t)$$

where $\xi_k \in L_G^p(\Omega_{t_k})$, $k = 0, 1, 2, \cdots, N-1$ are given. We denote these processes by $M_G^{p,0}(0, T)$.

Definition 2.8. *For any $\eta \in M_G^{p,0}(0, T)$, with the form $\eta_t(\omega) = \Sigma_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_k+1)}(t)$, the related Bochner integral on $[0, T]$ is defined :*

$$\int_0^T \eta_t(\omega) dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k)$$

We then complete the space $M_G^{p,0}(0, T)$ under norm $\| \cdot \|_M^p := \{\hat{\mathbb{E}}[\int_0^T | \cdot |^p dt]\}^{\frac{1}{p}}$ and denote the completion as $M_G^p(0, T)$. It is clear that $M_G^p(0, T) \subset M_G^q(0, T)$,if $1 \leq q \leq p$. Here is the definition of It$\hat{o}'s$ integral.

Definition 2.9. *For a $\eta \in M_G^{2,0}(0, T)$ of the form*

$$\eta_t(\omega) = \Sigma_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_k+1)}(t)$$

We define the itô integral as the following operator $I(\cdot) : M_G^{2,0}(0, T) \mapsto L_G^2(\Omega_T)$ as following:

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}),$$

where B_t is a G-Brownian Motion, and $G(\alpha) = \frac{1}{2}(\bar{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-)$, $0 \leq \underline{\sigma}^2 \leq \bar{\sigma}^2 \leq \infty$

Lemma 2.10. *For the mapping $I : M_G^{2,0}(0,T) \mapsto L_G^2(\Omega_T)$, we have:*

$$\hat{\mathbb{E}}[\int_0^T \eta_t dB_t] = 0, \quad (1)$$

$$\hat{\mathbb{E}}[|\int_0^T \eta_t dB_t|^2] \leq \bar{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_t|^2 dt], \quad (2)$$

Thus we can continuously extend I to a mapping from $M_G^2(0,T)$ to $L_G^2(\Omega_T)$, which is also denoted as I.

Definition 2.11. *For a $\eta \in M_G^2(0,T)$, the stochastic integral is defined as*

$$\begin{aligned}\int_0^T \eta_t dB_t &:= I(\eta),\\ \int_s^t \eta_u dB_u &:= \int_0^T I_{[s,t]}(u)\eta_u dB_u,\end{aligned}$$

Here is some basic properties of the integral of G-Brownian motion, and the proof is omitted.

Proposition 2.12. *For $\eta, \zeta \in M_G^2(0,T)$, and $0 \leq s \leq r \leq t \leq T$, we have:*
(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.
(ii) $\int_s^t (\alpha\eta_u + \zeta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \zeta_u dB_u$, *where α is bounded in $L_G^1(\Omega_s)$.*
(iii) $\hat{\mathbb{E}}[X + \int_r^T \eta_u dB_u | \Omega_s] = \hat{\mathbb{E}}[X|\Omega_s]$, *for $X \in L_G^1(\Omega_T)$*

Now we consider the quadratic variation process of G-Brownian motion $(B_t)_{t\geq 0}$ with $B_1 \stackrel{d}{=} N(\{0\} \times [\underline{\sigma}^2, \bar{\sigma}^2])$.

For a G-B.M. B_t and a partition Π_N of $[0,t]$: $0 = t_0 \leq t_1 \leq, \cdots, \leq t_N = t$, notice

$$B_t^2 = \sum_{j=0}^{N-1} 2B_{t_j^N}(B_{t_{j+1}^N} - B_{t_j^N}) + \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2.$$

As $||\Pi_N|| \to 0$, we can show that

$$\sum_{j=0}^{N-1} 2B_{t_j^N}(B_{t_{j+1}^N} - B_{t_j^N}) \stackrel{L_G^2(\Omega)}{\to} 2\int_0^t B_u dB_u,$$

so by the completeness of $L_G^2(\Omega)$, $\sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2$ must also converge in $L_G^2(\Omega)$.

Definition 2.13. *By the argument above, we define*

$$\begin{aligned}\langle B \rangle_t &:= \lim_{||\Pi_N|| \to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2\\ &= B_t^2 - 2\int_0^t B_r dB_r,\end{aligned}$$

and call $\langle B \rangle$ the quadratic variation process of G-Brownian Motion.

We now define the integral of a process $\eta \in M_G^1(0,T)$ with respect to $\langle B\rangle$. Firstly, we define a mapping:

$$Q_{0,T}(\eta) = \int_0^T \eta_t d\langle B\rangle_t := \sum_{j=0}^{N-1} \xi_j(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}) : M_G^{1,0}(0,T) \to L_G^1(\Omega_T).$$

Furthermore, we have the following lemma, which allows as to extend this mapping to $M_G^1(0,T)$.

Lemma 2.14. *For each $\eta \in M_G^{1,0}(0,T)$,*

$$\hat{\mathbb{E}}[|\int_0^T \eta_t d\langle B\rangle_t|] \leq \bar{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_t| dt].$$

Here two properties w.r.t $\langle B\rangle$ and its integral which would be used in the next part. The proof can be found in [6].

Proposition 2.15. (i) *for $\eta \in M_G^2(0,T)$, we have*

$$\hat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] = \hat{\mathbb{E}}[\int_0^T \eta_t^2 d\langle B\rangle_t].$$

(ii) *for each fixed $s, t \geq 0$, $\langle B\rangle_{t+s} - \langle B\rangle_s$ is identically distributed with $\langle B\rangle_t$ and independent from Ω_s. In addition, $\langle B\rangle_t$ is $N([\underline{\sigma}^2 t, \bar{\sigma}^2 t] \times 0) - distributed$.*

For the multi-dimensional case. Suppose $(B_t)_{t\geq 0}$ be a d-dimensional G-Brownian motion. For each $a \in \mathbb{R}^d$, we can easily check that $(B_t^a)_{t\geq 0} := (< B_t, a >)_{t\geq 0}$ is a 1-dimensional G_a-Brownian Motion, with $G_a(\alpha) = \frac{1}{2}(\sigma_{aa^T}^2 \alpha^+ - \sigma_{-aa^T}^2 \alpha^-)$, where $\sigma_{aa^T}^2 = 2G(aa^T)$ and $\sigma_{-aa^T}^2 = -2G(-aa^T)$. Then the related integrals with respect to B_t^a and $\langle B^a\rangle_t$ are same as the one dimensional case. Furthermore, for any $a, \bar{a} \in \mathbb{R}^d$, we define the mutual variation process by

$$\begin{aligned}
\langle B^a, B^{\bar{a}}\rangle_t &:= \frac{1}{4}[\langle B^a + B^{\bar{a}}\rangle_t - \langle B^a - B^{\bar{a}}\rangle_t] \\
&= \frac{1}{4}[\langle B^{a+\bar{a}}\rangle_t - \langle B^{a-\bar{a}}\rangle_t].
\end{aligned}$$

We can prove that for a sequence of partition $\pi_t^n, n = 1, 2, \cdots,$ of $[0,t]$ with $||\pi_t^n|| \to 0$, we have

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} (B_{t_{k+1}^n}^a - B_{t_k^n}^a)(B_{t_{k+1}^n}^{\bar{a}} - B_{t_k^n}^{\bar{a}}) = \langle B^a, B^{\bar{a}}\rangle_t.$$

Now we give the famous $It\hat{o}'s$ formula in G-framework. More recent progresses can be read from [3].

Theorem 2.16. *(G-Itô′s Formula) Let Φ be a twice continuous function on $\mathbb{R}^n$ with polynomial growth growth for the first and second order derivatives. X is a Itô process, i.e.*

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^{\nu ij} d\langle B^i, B^j\rangle_+ \int_0^t \beta_s^{\nu j} dB_s^j$$

where $\nu = 1, ..., n,\ i, j = 1, ..., d,\ \alpha_s^\nu, \eta_s^{\nu ij}, \beta_s^{\nu j}$ *are bounded processes in* $M_G^2(0,T)$. *Here repeated indices means summation over the same indices. Then for each* $t \geq s \geq 0$ *we have in* $L_G^2(\Omega_t)$:

$$\begin{aligned}\Phi(X_t) - \Phi(X_s) &= \int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha_u^\nu du \\ &+ \int_s^t [\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}] d\langle B^i, B^j\rangle_u\end{aligned}$$

3 Complex Sublinear Expectation

Here we try to define a sublinear expectation under complex case, which means we have to decide what's "sublinear" here. From here on, when we compare two complex numbers, we compare them as real vectors. What we do here is to connect the G-expectation with complex analysis, so we define complex sublinear expectation in the following heuristic way.

Given a set Ω and a linear space $\mathcal{H}$ consisting of real valued functions on Ω, we suppose $\mathcal{H}$ is a vector lattice and consider the following set of functions on Ω:

$$\mathcal{H}_\mathbb{C} = \{X + iY | X, Y \in \mathcal{H}\},\ where\, i^2 = -1$$

Definition 3.1. *Suppose* $(\Omega\ \mathcal{H}\ \hat{\mathbb{E}})$*is a sublinear space, we define a function* $\hat{\mathbb{E}}_\mathbb{C}$ *from* $\mathcal{H}_\mathbb{C}$ *to complex field as:*

$$\begin{aligned}\hat{\mathbb{E}}_\mathbb{C}: \qquad \mathcal{H}_\mathbb{C} &\mapsto \mathbb{C} \\ X + iY &\mapsto \hat{\mathbb{E}}[X] + i\hat{\mathbb{E}}[Y]\end{aligned}$$

Remark 3.2. *Notice that for any* $Z \in \mathcal{H}_\mathbb{C}$, Z *has a unique expression as* $X+iY$, *so* $\hat{\mathbb{E}}_\mathbb{C}$ *in well defined.*

Here are some basic properties of $\hat{\mathbb{E}}_\mathbb{C}$*:*

(i) *constant preserving:*

$$\hat{\mathbb{E}}_\mathbb{C}(z) = z, \qquad \forall\ z \in \mathbb{C}$$

(ii) *positive homogeneity:*

$$\hat{\mathbb{E}}_\mathbb{C}(\lambda Z) = \lambda\hat{\mathbb{E}}_\mathbb{C}(Z), \qquad \lambda \geq 0 \quad Z \in \mathcal{H}_\mathbb{C}$$

(iii) *constant transferability:*

$$\hat{\mathbb{E}}_\mathbb{C}(Z + c) = \hat{\mathbb{E}}_\mathbb{C}(Z) + c, \qquad c \in \mathbb{C} \quad Z \in \mathcal{H}_\mathbb{C}$$

(iv) *monotonicity:*

$$\hat{\mathbb{E}}_\mathbb{C}(Z_1) \geq \hat{\mathbb{E}}_\mathbb{C}(Z_2) \qquad if\ Z_1 \geq Z_2$$

(v) *convexity:*

$$\hat{\mathbb{E}}_{\mathbb{C}}[\alpha Z_1 + (1-\alpha)Z_2] \le \alpha\hat{\mathbb{E}}_{\mathbb{C}}(Z_1) + (1-\alpha)\hat{\mathbb{E}}_{\mathbb{C}}(Z_2), \qquad \alpha \in [0,1]$$

Remark 3.3. *Conversely, given some properties of the complex expectation, we can define the sublinear expectation in a more general version:*

$$\hat{\mathbb{E}}_{\mathbb{C}} : \mathcal{H}_{\mathbb{C}} \to \mathbb{C}$$

satisfying

$$\hat{\mathbb{E}}_{\mathbb{C}}[X] \in \mathbb{R} \quad for\, any\, X \in \mathcal{H}$$

and

$$\begin{array}{rc} (i) & \hat{\mathbb{E}}_{\mathbb{C}}(c) = c,\ \forall c \in \mathbb{C} \\ (ii) & \hat{\mathbb{E}}_{\mathbb{C}}(Z_1) \ge \hat{\mathbb{E}}_{\mathbb{C}}(Z_2),\ Z_1 \ge Z_2,\ Z_1,\ Z_2 \in \mathcal{H}_{\mathbb{C}} \\ (iii) & \hat{\mathbb{E}}_{\mathbb{C}}(Z_1+Z_2) \le \hat{\mathbb{E}}_{\mathbb{C}}(Z_1) + \hat{\mathbb{E}}_{\mathbb{C}}(Z_2) \\ (iv) & \hat{\mathbb{E}}_{\mathbb{C}}(\lambda Z) = \lambda\hat{\mathbb{E}}_{\mathbb{C}}(Z),\ \lambda \ge 0\ \lambda \in \mathbb{R} \\ (v) & \hat{\mathbb{E}}_{\mathbb{C}}(Z) \le \hat{\mathbb{E}}_{\mathbb{C}}(X) + i\hat{\mathbb{E}}_{\mathbb{C}}(Y),\ Z = X + iY \end{array}$$

If (v) in the above is changed to an equation, we get the same sublinear expectation as definition (3.1) by the representation theorem in the real case.

Remark 3.4. *For each $\varphi \in C_{l.lip}(\mathbb{C})$, we suppose $\varphi(x+iy) = \varphi_1(x,y) + i\varphi_2(x,y)$. Then we have $\varphi_i(x,y) \in C_{l.lip}(\mathbb{R}^2)$. Also, if $\varphi_i(x,y) \in C_{l.lip}(\mathbb{R}^2)$, $i = 1,2$, we have $\varphi(x+iy)(:= \varphi_1(x,y) + i\varphi_2(x,y))$ belongs to $C_{l.lip}(\mathbb{C})$.*

Definition 3.5. *Suppose $Z = (Z_1, \cdots, Z_n)$ a n-dimensional vector on the complex sublinear space $(\Omega,\ \mathcal{H}_{\mathbb{C}},\ \hat{\mathbb{E}}_{\mathbb{C}})$. The functional from $C_{l.lip}(\mathbb{C}^n)$ to $\mathbb{C}$ defined by*

$$\begin{aligned} \mathbb{F}_Z(\varphi) &:= \hat{\mathbb{E}}_{\mathbb{C}}[\varphi(Z)] \\ &= \mathbb{E}[\varphi_1(X,Y)] + \mathbb{E}[\varphi_2(X,Y)], \end{aligned} \tag{3}$$

where $\varphi \in C_{l.lip}(\mathbb{C}^n)$ and $\varphi(x+iy) := \varphi_1(x,y) + i\varphi_2(x,y)$, is called the distribution of Z under $\hat{\mathbb{E}}_{\mathbb{C}}$. Also, the triple $(\mathbb{C}^n,\ C_{l.lip}(\mathbb{C}^n),\ \mathbb{F}_Z)$ forms a complex sublinear space.

Definition 3.6. *Two n-dimensional random vectors Z_1 and Z_2 defined on two complex sublinear expectation spaces $(\Omega_1,\ \mathcal{H}^1_{\mathbb{C}},\ \hat{\mathbb{E}}^1_{\mathbb{C}})$ and $(\Omega_2,\ \mathcal{H}^2_{\mathbb{C}},\ \hat{\mathbb{E}}^2_{\mathbb{C}})$ respectively, are called identically distributed if*

$$\hat{\mathbb{E}}^1_{\mathbb{C}}[\varphi(Z_1)] = \hat{\mathbb{E}}^2_{\mathbb{C}}[\varphi(Z_2)], \qquad for\, any\, \varphi \in C_{l.lip}(\mathbb{C}^n).$$

Such relation is denoted by $Z_1 \overset{d}{=} Z_2$.

Proposition 3.7. *Let* $(\Omega, \mathcal{H}_{\mathbb{C}}, \hat{\mathbb{E}}_{\mathbb{C}})$ *be a complex sublinear expectation.* Z_1, Z_2 *are two random variables and* Z_2 *satisfies*

$$\hat{\mathbb{E}}_{\mathbb{C}}[Z_2] = -\hat{\mathbb{E}}_{\mathbb{C}}[-Z_2]$$

Then we have

$$\hat{\mathbb{E}}_{\mathbb{C}}[Z_1 + cZ_2] = \hat{\mathbb{E}}_{\mathbb{C}}[Z_1] + c\ \hat{\mathbb{E}}_{\mathbb{C}}[Z_2], \quad for\, any\, c \in \mathbb{C}.$$

Proof. Suppose $Z_1 = X_1 + iY_1,\ Z_2 = X_2 + iY_2,\ c = x + iy$, where $X_i,\ Y_i \in \mathcal{H},\ and\ x,\ y \in \mathbb{R}$. Since $\hat{\mathbb{E}}_{\mathbb{C}}[Z_2] = -\hat{\mathbb{E}}_{\mathbb{C}}[-Z_2]$, we have

$$\mathbb{E}[X_2] = -\mathbb{E}[-X_2], \quad \mathbb{E}[Y_2] = -\mathbb{E}[-Y_2]$$

This leads to

$$\hat{\mathbb{E}}[X + aX_2] = \hat{\mathbb{E}}[X] + a\ \hat{\mathbb{E}}[X_2], for\, a \in \mathbb{R}, X \in \mathcal{H}$$

and similar result for Y_2.

$$\begin{aligned}
\hat{\mathbb{E}}_{\mathbb{C}}[Z_1 + cZ_2] &= \hat{\mathbb{E}}_{\mathbb{C}}[(X_1 + xX_2 - yY_2) + i(Y_1 + xY_2 + yX_2)] \\
&= \hat{\mathbb{E}}[X_1] + x\hat{\mathbb{E}}[X_2] - y\hat{\mathbb{E}}[Y_2] + i\hat{\mathbb{E}}[Y_1] + ix\hat{\mathbb{E}}[Y_2] + iy\hat{\mathbb{E}}[X_2] \\
&= \hat{\mathbb{E}}_{\mathbb{C}}[Z_1] + c\hat{\mathbb{E}}_{\mathbb{C}}[Z_2]
\end{aligned}$$

□

Another important definition in sublinear expectation is the independence. Since our main purpose is to connect real sublinear expectation with complex analysis, we find the following definition does this job.

Definition 3.8. *In a complex sublinear space* $(\Omega, \mathcal{H}_{\mathbb{C}}, \hat{\mathbb{E}}_{\mathbb{C}})$*, a random vector* $Z_2 \in \mathcal{H}_{\mathbb{C}}^n$ *is said to be independent from another one* $Z_1 \in \mathcal{H}_{\mathbb{C}}^m$ *if for each test function* $\varphi \in C_{l.lip}(\mathbb{C}^{m+n})$ *we have*

$$\hat{\mathbb{E}}_{\mathbb{C}}[\varphi(Z_1, Z_2)] = \hat{\mathbb{E}}_{\mathbb{C}}[\hat{\mathbb{E}}_{\mathbb{C}}[\varphi(z, Z_2)]_{z=Z_1}]$$

Here is an important lemma to connect the complex case with the real one.

Lemma 3.9. *Let* $Z_1 = X_1 + iY_1,\ Z_2 = X_2 + iY_2$*, where* $X_1,\ Y_1 \in \mathcal{H}^n,\ X_2,\ Y_2 \in \mathcal{H}^m$*. Then* Z_2 *is independent of* Z_1 *if and only if* $(X_2,\ Y_2)$ *is independent of* $(X_1,\ Y_1)$ *under* $\hat{\mathbb{E}}$.

Proof. If Z_2 is independent of Z_1, we need to show that for any $\varphi \in C_{l.lip}(\mathbb{R}^{2n+2m})$ we have

$$\hat{\mathbb{E}}[\varphi(X_1, Y_1, X_2, Y_2)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, y, X_2, Y_2)]_{(x,y)=(X_1,Y_1)}]$$

For any fixed $\varphi \in C_{l.lip}(\mathbb{R}^{2n+2m})$, we take $\psi := \varphi + i \cdot 0 = \varphi$, which belongs to $C_{l.lip}(\mathbb{C}^{n+m})$.

By the definition of independence under complex case, we have

$$\begin{aligned}\hat{\mathbb{E}}[\varphi(X_1,Y_1,X_2,Y_2)] &= \hat{\mathbb{E}}_{\mathbb{C}}[\psi(Z_1,Z_2)] \\ &= \hat{\mathbb{E}}_{\mathbb{C}}[\hat{\mathbb{E}}_{\mathbb{C}}[\psi(z,Z_2)]_{z=Z_1}] \\ &= \hat{\mathbb{E}}_{\mathbb{C}}[\hat{\mathbb{E}}[\varphi(x,y,X_2,Y_2)]_{(x,y)=(X_1,Y_1)}] \\ &= \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,y,X_2,Y_2)]_{(x,y)=(X_1,Y_1)}]\end{aligned}$$

For the backward implication, if $\varphi \in C_{l.lip}(\mathbb{C}^{m+n})$, we suppose $\varphi = \varphi_1 + i\varphi_2,\ where \varphi_1,\ \varphi_2 \in C_{l.lip}(\mathbb{R}^{2n+2m})$

$$\begin{aligned}\hat{\mathbb{E}}_{\mathbb{C}}[\varphi(Z_1,Z_2)] &= \hat{\mathbb{E}}[\varphi_1(X_1,Y_1,X_2,Y_2)] + i\hat{\mathbb{E}}[\varphi_2(X_1,Y_1,X_2,Y_2)] \\ &= \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi_1(x,y,X_2,Y_2)]_{(x,y)=(X_1,Y_1)}] + i\hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi_2(x,y,X_2,Y_2)]_{(x,y)=(X_1,Y_1)}] \\ &= \hat{\mathbb{E}}_{\mathbb{C}}[\hat{\mathbb{E}}[\varphi_1(x,y,X_2,Y_2)]_{(x,y)=(X_1,Y_1)} + i\hat{\mathbb{E}}[\varphi_2(x,y,X_2,Y_2)]_{(x,y)=(X_1,Y_1)}] \\ &= \hat{\mathbb{E}}_{\mathbb{C}}[\hat{\mathbb{E}}_{\mathbb{C}}[\varphi(z,Z_2)]_{z=Z_1}]\end{aligned}$$

□

Remark 3.10. *(Completion of Complex Sublinear Expectation Space) For a real sublinear expectation space* $(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}})$*, we denote* $(\Omega, \mathcal{H}^p, \hat{\mathbb{E}})$ *the completion of the real sublinear expectation space under norm* $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$*. Then we define* $\mathcal{H}^p_{\mathbb{C}} = \{X + iY \mid X, Y \in \hat{\mathcal{H}}^p\}$*, and it is also a banach space under norm* $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$ *by the completeness of* $\hat{\mathcal{H}}^p$*.*

4 Complex G-normal distribution and G-Brownian Motion

Here we define the complex G-normal distribution and G-Brownian Motion.

Definition 4.1. *(Complex G-normal distribution) A d-dimensional random variable* $Z = (Z_1, \cdots, Z_d)$ *on a complex sublinear expectation space* $(\Omega,\ \mathcal{H}_{\mathbb{C}},\ \hat{\mathbb{E}}_{\mathbb{C}})$ *is called complex G-normal distributed if*

$$aZ + b\hat{Z} \overset{d}{=} \sqrt{a^2+b^2}Z, for\ a,b \geq 0$$

where $\hat{Z}$ *is an independent copy of* Z*.*

Example 4.2. *Suppose Z is a complex G-normal distributed variable. By the definition of the complex G-normal distribution,* $e^{i\theta}Z$ *is also normal distributed, which means a change of the argument of a complex G-normal distributed variable is also G-normal distributed.*

Remark 4.3. *By the definition of identically distributed and lemma(3.9),* $Z = X + iY$ *is complex G-normal distributed if and only if* (X,Y) *is G-normal distributed in* $(\Omega,\ \mathcal{H},\ \hat{\mathbb{E}})$

Remark 4.4. *If we define* $\omega(t,z) := \hat{\mathbb{E}}_{\mathbb{C}}[\varphi(z+\sqrt{t}Z)]$, *where* $\varphi \in C_{b.lip}(\mathbb{C}^d)$, $Z \in \mathcal{H}_{\mathbb{C}}^d$ *and* Z *is complex G-normal distributed, we have*

$$\begin{aligned}
\omega(t+s,z) &= \hat{\mathbb{E}}_{\mathbb{C}}[\varphi(z+\sqrt{t+s}Z)] \\
&= \hat{\mathbb{E}}_{\mathbb{C}}[\varphi(z+\sqrt{s}Z+\sqrt{t}\hat{Z})] \\
&= \hat{\mathbb{E}}_{\mathbb{C}}[\hat{\mathbb{E}}_{\mathbb{C}}[\varphi(z+\sqrt{s}c+\sqrt{t}Z]_{c=Z}] \\
&= \hat{\mathbb{E}}_{\mathbb{C}}[\omega(t,z+\sqrt{s}Z)]
\end{aligned}$$

If $\omega(t,z)$ *is continuously differentiable on* t, *complex differentiable on* z, *and has at most polynomially growth at infinity, thanks to proposition (3.7), we could get a* $G_{\mathbb{C}}$ *heat equation:*

$$\partial_t \omega(t,z) - G_{\mathbb{C}}(D_2\omega(t,z)) = 0$$

where $G_{\mathbb{C}}(c) = \frac{1}{2}\hat{\mathbb{E}}_{\mathbb{C}}[cZ^2]$. *Here we should notice that* G, *which has 4 parameters, could determine* $G_{\mathbb{C}}$, *a 3-parameter function and the converse is not usually true. However, there are conditions under which* G *and* $G_{\mathbb{C}}$ *are mutually determined, such as* $Z = X + i\hat{X}$, *where* $\hat{X}$ *is the independent copy of the* $\tilde{G}$ *normal distributed variable* X.

Now we can turn to stochastic analysis. Let's start with complex G-Brownian Motion.

Definition 4.5. *complex G-Brownian Motion A d-dimensional process* $(B_t)_{t\geq 0}$ *is called a complex G-Brownian Motion if the following are satisfied:*
(i) $B_0(\omega) = 0$
(ii) *For each* $s,t \geq 0$, *the increment* $B_{t+s} - B_t$ *is complex G-normal distributed and is independent from* $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, *for any* $n \in \mathbb{N}$ *and* $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$.

5 Stochastic Integral and Related Stochastic Calculus

Definition 5.1. *For each* $T \in [0,\infty)$, *we set Lipschitz cylinder functions as:*

$$Lip_{\mathbb{C}}(\Omega_T) := \{\varphi(B_{t_1\wedge T}, B_{t_2\wedge T}, \cdots, B_{t_n\wedge T}) : n \in \mathbb{N},\ t_1,\cdots,t_n \in [0,\infty),\ \varphi \in C_{l.lip}(\mathbb{C}^{d\times n})\}$$

where $(B_t)_{t\geq 0}$ *is a complex G-Brownian Motion. It is clear that* $Lip_{\mathbb{C}}(\Omega_t) \subseteq Lip_{\mathbb{C}}(\Omega_T)$ *for* $t \leq T$. *Also, we let* $Lip_{\mathbb{C}}(\Omega) := \bigcup_{n=1}^{\infty} Lip_{\mathbb{C}}(\Omega_n)$. *Then we have a complex sublinear expectation called complex G-expectation as:*

$$\hat{\mathbb{E}}_{\mathbb{C}}[\cdot] \ : \ Lip_{\mathbb{C}}(\Omega) \longmapsto \mathbb{C}.$$

Remark 5.2. *The corresponding conditional expectation also explains itself as the real case.*

To complete $Lip_{\mathbb{C}}(\Omega_T)$, we need a small lemma to define a norm on it.

Lemma 5.3. *$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is a real sublinear space. Then $\|\cdot\| := (\hat{\mathbb{E}}|\cdot|^p)^{\frac{1}{p}}$, $p > 1$, defines a norm on $(\Omega, \mathcal{H}_{\mathbb{C}}, \hat{\mathbb{E}}_{\mathbb{C}})$.*

Proof. The proof is similar as the classical one, so we omit it.

□

Remark 5.4. *We define a norm $\hat{\mathbb{E}}[|\cdot|^p]^{\frac{1}{p}}$ on $Lip_{\mathbb{C}}(\Omega)$ and denote $L_G^p(\Omega)_{\mathbb{C}}$ as the completion of linear space $Lip_{\mathbb{C}}(\Omega)$. Also, $L_G^p(\Omega_T)_{\mathbb{C}}$ explains itself. Then we have the following.*

Proposition 5.5. *The completion $L_G^p(\Omega_T)_{\mathbb{C}}$ of $Lip_{\mathbb{C}}(\Omega_T)$ under norm $\hat{\mathbb{E}}[|\cdot|^p]^{\frac{1}{p}}$ have the following expression:*

$$L_G^p(\Omega_T)_{\mathbb{C}} = \{\xi_1 + i\xi_2 |\ \xi_1, \xi_2 \in L_G^p(\Omega_T)\}$$

under norm $\hat{\mathbb{E}}[|\cdot|^p]^{\frac{1}{p}}$, where $L_G^p(\Omega_T)$ is the completion of

$$Lip(\Omega_T) := \{\varphi(B^{(1)}_{t_1\wedge T}, B^{(1)}_{t_2\wedge T}, \cdots, B^{(1)}_{t_n\wedge T}, B^{(2)}_{t_1\wedge T}, \cdots, B^{(2)}_{t_n\wedge T}) | n \in \mathbb{N}, \varphi \in C_{l.lip}(\mathbb{R}^{d\times 2n})\}$$

and $B_t = B_t^{(1)} + iB_t^{(2)}$ is the complex G-Brownian Motion.

Proof. For any $\xi_1, \xi_2 \in Lip(\Omega_T)$, without loss of generality, we can suppose they have the same form as $\xi_i = \varphi_i(B^{(1)}_{t_1\wedge T}, B^{(1)}_{t_2\wedge T}, \cdots, B^{(1)}_{t_n\wedge T}, B^{(2)}_{t_1\wedge T}, \cdots, B^{(2)}_{t_n\wedge T})$, where $\varphi_i \in C_{l.lip}(\mathbb{R}^{d\times 2n}), i = 1, 2$. Then we can take $\varphi = \varphi_1 + i\varphi_2$ and conclude that $\xi_1 + i\xi_2 \in L_G^p(\Omega_T)_{\mathbb{C}}$. The forward inclusion follows from facts (i) $Lip(\Omega_T)$ is dense in $L_G^p(\Omega_T)$ and (ii) $L_G^p(\Omega_T)_{\mathbb{C}}$ is complete.

For the converse inclusion, notice that for any $\varphi \in C_{l.lip}(\mathbb{C}^{d\times n})$, φ has a unique decomposition $\varphi = \varphi_1 + i\varphi_2$ where $\varphi_1, \varphi_2 \in C_{l.lip}(\mathbb{R}^{2d\times n})$. It means $Lip_{\mathbb{C}}(\Omega_T) \subseteq \{\xi_1 + i\xi_2 | \xi_1, \xi_2 \in L_G^p(\Omega_T)\}$ by the completeness of $\{\xi_1 + i\xi_2 | \xi_1, \xi_2 \in L_G^p(\Omega_T)\}$. □

Now we can define $it\hat{o}'s$ integral by starting from simple processes as the following: for a partition $\Pi = \{t_0, t_1, \cdots, t_N\}$ of $[0, T]$, we define

$$\eta_t(\omega) := \Sigma_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t)$$

where $\xi_k \in L_G^p(\Omega_{t_k})_{\mathbb{C}}$, $k = 0, 1, 2, \cdots, N-1$ are given. We denote these processes by $M_G^{p,0}(0, T)_{\mathbb{C}}$.

Definition 5.6. *For any $\eta \in M_G^{p,0}(0, T)_{\mathbb{C}}$, with the form $\eta_t(\omega) = \Sigma_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t)$, the related Bochner integral on $[0, T]$ is defined as natural as:*

$$\int_0^T \eta_t(\omega) dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k)$$

Then we can complete the space $M_G^{p,0}(0,T)_{\mathbb{C}}$ under norm $\|\cdot\|_M^p := \{\hat{\mathbb{E}}[\int_0^T |\cdot|^p dt]\}^{\frac{1}{p}}$ and denote the completion as $M_G^p(0,T)_{\mathbb{C}}$. Similar as $L_G^p(\Omega_T)_{\mathbb{C}}$, $M_G^p(0,T)_{\mathbb{C}}$ has the expression:

$$M_G^p(0,T)_{\mathbb{C}} = \{\eta_t^{(1)} + i\eta_t^{(2)} \mid \eta_t^{(1)}, \eta_t^{(2)} \in M_G^p(0,T)\}$$

where $M_G^p(0,T)$ is the corresponding space in real case.

Definition 5.7. *For a $\eta \in M_G^{2,0}(0,T)_{\mathbb{C}}$ of the form*

$$\eta_t(\omega) = \Sigma_{k=0}^{N-1}\xi_k(\omega)I_{[t_k,t_k+1)}(t)$$

We define the itô integral as the following operator $I(\cdot) : M_G^{2,0}(0,T)_{\mathbb{C}} \mapsto L_G^2(\Omega_T)_{\mathbb{C}}$ as following:

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}),$$

where $B_t = B_t^{(1)} + iB_t^{(2)}$ is a Brownian Motion.

Proposition 5.8. *For the mapping $I : M_G^{2,0}(0,T)_{\mathbb{C}} \mapsto L_G^2(\Omega_T)_{\mathbb{C}}$, we have:*

$$\hat{\mathbb{E}}_{\mathbb{C}}[\int_0^T \eta_t dB_t] = 0, \tag{4}$$

$$\hat{\mathbb{E}}[|\int_0^T \eta_t dB_t|^2] \leq 16K\hat{\mathbb{E}}[\int_0^T |\eta_t|^2 dt], \tag{5}$$

where $K = (\bar{\sigma}_1^2 + \bar{\sigma}_2^2)$, $\bar{\sigma}_1^2 := \hat{\mathbb{E}}[(B_1^{(1)})^2]$, $\bar{\sigma}_2^2 := \hat{\mathbb{E}}[(B_1^{(2)})^2]$.

Proof. We denote

$$\triangle B_j = B_{t_{j+1}} - B_{t_j},$$

$$\triangle B_j^{(i)} = B_{t_{j+1}}^{(i)} - B_{t_j}^{(i)}, i = 1, 2$$

For (4), we notice $\hat{\mathbb{E}}_{\mathbb{C}}[\int_0^T \eta_t dB_t] = \hat{\mathbb{E}}_{\mathbb{C}}[\sum_{j=0}^{N-1} \xi_j \triangle B_j]$, and

$$\begin{aligned}\hat{\mathbb{E}}_{\mathbb{C}}[\xi_j \triangle B_j] &= \hat{\mathbb{E}}_{\mathbb{C}}[(\xi_j^{(1)} + i\xi_j^{(2)})(\triangle B_j^{(1)} + i\triangle B_j^{(2)})] \\ &= \hat{\mathbb{E}}[\xi_j^{(1)} \triangle B_j^{(1)} - \xi_j^{(2)} \triangle B_j^{(2)}] + i\hat{\mathbb{E}}[\xi_j^{(1)} \triangle B_j^{(2)} + \xi_j^{(2)} \triangle B_j^{(1)}].\end{aligned}$$

Also, we have $\hat{\mathbb{E}}[\xi_j^{(i)} \triangle B_j^{(k)}] = 0, i, k = 1, 2$. By proposition (3.7), these equations imply the conclusion.

For (5), firstly notice

$$
\begin{aligned}
\hat{\mathbb{E}}[|\int_0^T \eta_t dB_t|^2] &= \hat{\mathbb{E}}[|\sum_{j=0}^{N-1}[(\xi_j^{(1)}\triangle B_j^{(1)} - \xi_j^{(2)}\triangle B_j^{(2)}) + i(\xi_j^{(1)}\triangle B_j^{(2)} + \xi_j^{(2)}\triangle B_j^{(1)})]|^2] \\
&\leq \hat{\mathbb{E}}[(\sum_{j=0}^{N-1}(\xi_j^{(1)}\triangle B_j^{(1)} - \xi_j^{(2)}\triangle B_j^{(2)}))^2] + \hat{\mathbb{E}}[(\sum_{j=0}^{N-1}(\xi_j^{(1)}\triangle B_j^{(2)} + \xi_j^{(2)}\triangle B_j^{(1)}))^2] \\
&\leq 2[\hat{\mathbb{E}}(\sum_{j=0}^{N-1}(\xi_j^{(1)}\triangle B_j^{(1)}))^2 + \hat{\mathbb{E}}(\sum_{j=0}^{N-1}(\xi_j^{(2)}\triangle B_j^{(2)}))^2 + \hat{\mathbb{E}}(\sum_{j=0}^{N-1}(\xi_j^{(1)}\triangle B_j^{(2)}))^2 \\
&+ \hat{\mathbb{E}}(\sum_{j=0}^{N-1}(\xi_j^{(2)}\triangle B_j^{(1)}))^2]
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\hat{\mathbb{E}}[\xi_j^{(i)}\triangle B_j^{(k)}\xi_l^{(m)}\triangle B_l^{(n)}] &= \hat{\mathbb{E}}[\xi_j^{(i)}\triangle B_j^{(k)}\xi_l^{(m)}\hat{\mathbb{E}}[\triangle B_l^{(n)}|\Omega_{t_l}]] \\
&= 0,
\end{aligned}
$$

where $l \geq j, and i, k, m, n = 1, 2,$
and

$$
\int_0^T |\eta_t|^2 dt = \sum_{j=0}^{N-1}((\xi_j^{(1)})^2 + (\xi_j^{(2)})^2)(t_{j+1} - t_j).
$$

We only need to show

$$
\hat{\mathbb{E}}[[\sum_{j=0}^{N-1}(\xi_j^{(1)}\triangle B_j^{(1)})]^2] \leq 2K\hat{\mathbb{E}}[\sum_{j=0}^{N-1}((\xi_j^{(1)})^2 + (\xi_j^{(2)})^2)(t_{j+1} - t_j)],
$$

and the others are similar.
Notice that

$$
\begin{aligned}
\hat{\mathbb{E}}[(\xi_j^{(1)})^2(\triangle B_j^{(1)})^2 + (\xi_{j+1}^{(1)})^2(\triangle B_{j+1}^{(1)})^2] &= \hat{\mathbb{E}}[(\xi_j^{(1)})^2(\triangle B_j^{(1)})^2 + (\xi_{j+1}^{(1)})^2\hat{\mathbb{E}}[(\triangle B_{j+1}^{(1)})^2|\Omega_{t_j+1}]] \\
&= \hat{\mathbb{E}}[(\xi_j^{(1)})^2(\triangle B_j^{(1)})^2 + \bar{\sigma}_1^2(\xi_{j+1}^{(1)})^2(t_{j+2} - t_{j+1})] \\
&= \hat{\mathbb{E}}[\hat{\mathbb{E}}[(\xi_j^{(1)})^2(\triangle B_j^{(1)})^2 + \bar{\sigma}_1^2(\xi_{j+1}^{(1)})^2(t_{j+2} - t_{j+1})]|\Omega_{t_j}] \\
&\leq \hat{\mathbb{E}}[\hat{\mathbb{E}}[(\xi_j^{(1)})^2(\triangle B_j^{(1)})^2|\Omega_{t_j}] + \hat{\mathbb{E}}[\bar{\sigma}_1^2(\xi_{j+1}^{(1)})^2(t_{j+2} - t_{j+1})|\Omega_{t_j}]] \\
&= \hat{\mathbb{E}}[(\xi_j^{(1)})^2\bar{\sigma}_1^2(t_{j+1} - t_j) + \hat{\mathbb{E}}[\bar{\sigma}_1^2(\xi_{j+1}^{(1)})^2(t_{j+2} - t_{j+1})|\Omega_{t_j}]] \\
&= \hat{\mathbb{E}}[\bar{\sigma}_1^2(\xi_j^{(1)})^2(t_{j+1} - t_j) + \bar{\sigma}_1^2(\xi_{j+1}^{(1)})^2(t_{j+2} - t_{j+1})].
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\hat{\mathbb{E}}[[\sum_{j=0}^{N-1}(\xi_j^{(1)}\triangle B_j^{(1)})]^2] &\leq 2\hat{\mathbb{E}}[\sum_{j=0}^{N-1}(\xi_j^{(1)})^2(\triangle B_j^{(1)})^2] + 2\hat{\mathbb{E}}[\sum_{j=0}^{N-1}(\xi_j^{(2)})^2(\triangle B_j^{(2)})^2] \\
&\leq 2\bar{\sigma}_1^2\hat{\mathbb{E}}[\sum_{j=0}^{N-1}[((\xi_j^{(1)})^2+(\xi_j^{(2)})^2)(t_{j+1}-t_j)]] \\
&+ 2\bar{\sigma}_2^2\hat{\mathbb{E}}[\sum_{j=0}^{N-1}[((\xi_j^{(1)})^2+(\xi_j^{(2)})^2)(t_{j+1}-t_j)]] \\
&\leq 2K\hat{\mathbb{E}}[\sum_{j=0}^{N-1}[((\xi_j^{(1)})^2+(\xi_j^{(2)})^2)(t_{j+1}-t_j)]]
\end{aligned}
$$

□

So we can extend the stochastic integral mapping I from $M_G^{2,0}(0,T)_{\mathbb{C}}$ to $M_G^2(0,T)_{\mathbb{C}}$.

Definition 5.9. *For a $\eta \in M_G^2(0,T)_{\mathbb{C}}$, the stochastic integral is defined as*

$$
\begin{aligned}
\int_0^T \eta_t dB_t &:= I(\eta), \\
\int_s^t \eta_u dB_u &:= \int_0^T I_{[s,t]}(u)\eta_u dB_u,
\end{aligned}
$$

where I is the extension of the former integral mapping from $M_G^2(0,T)_{\mathbb{C}}$ to $L_G^2(\Omega_T)_{\mathbb{C}}$ by Hahn-Banach extension theorem.

Naturally, we have the following basic properties. The proof is trivial.

Proposition 5.10. *For $\eta, \zeta \in M_G^2(0,T)_{\mathbb{C}}$, and $0 \leq s \leq r \leq t \leq T$, we have:*
(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.
(ii) $\int_s^t (\alpha\eta_u + \zeta_u)dB_u = \alpha\int_s^t \eta_u dB_u + \int_s^t \zeta_u dB_u$, *where α is bounded in $L_G^2(\Omega_s)_{\mathbb{C}}$.*
(iii) $\hat{\mathbb{E}}_{\mathbb{C}}[Z + \int_r^T \eta_u dB_u|\Omega_s] = \hat{\mathbb{E}}_{\mathbb{C}}[Z|\Omega_s]$, *for $Z \in L_G^2(\Omega_T)_{\mathbb{C}}$*

Now we turn to Quadratic Variation Process of complex G-Brownian Motion. For a complex G-B.M. B_t, notice

$$
B_t^2 = \sum_{j=0}^{N-1} 2B_{t_j^N}(B_{t_{j+1}^N} - B_{t_j^N}) + \sum_{j=0}^{N-1}(B_{t_{j+1}^N} - B_{t_j^N})^2
$$

for a partition Π_N of $[0,t]$: $0 = t_0 \leq t_1 \leq, \cdots, \leq t_N = t$. As $||\Pi_N|| \to 0$, by independence under complex framework, we can show that

$$
\sum_{j=0}^{N-1} 2B_{t_j^N}(B_{t_{j+1}^N} - B_{t_j^N}) \overset{L_G^2(\Omega)_{\mathbb{C}}}{\to} 2\int_0^t B_u dB_u,
$$

so by the completeness of $L_G^2(\Omega)_{\mathbb{C}}$, $\sum_{j=0}^{N-1}(B_{t_{j+1}^N} - B_{t_j^N})^2$ must also converge in $L_G^2(\Omega)_{\mathbb{C}}$.

Definition 5.11. *By the argument above, we define*

$$\begin{aligned}\langle B\rangle_t &:= \lim_{||\Pi_N||\to 0}\sum_{j=0}^{N-1}(B_{t_{j+1}^N} - B_{t_j^N})^2 \\ &= B_t^2 - 2\int_0^t B_r dB_r,\end{aligned}$$

and call $\langle B\rangle$ the quadratic variation process of complex G-Brownian Motion.

Remark 5.12. *Since $L_G^p(\Omega_T)_{\mathbb{C}} = \{\xi_1 + i\xi_2 |\ \xi_1, \xi_2 \in L_G^p(\Omega_T)\}$, we can similarly show that*

$$\begin{aligned}\int_0^T \eta_t dB_t &= \int_0^T (\eta_t^{(1)} + i\eta_t^{(2)}) d(B_t^{(1)} + iB_t^{(2)}) \\ &= [\int_0^T \eta_t^{(1)} dB_t^{(1)} - \int_0^T \eta_t^{(2)} dB_t^{(2)}] + i[\int_0^T \eta_t^{(1)} dB_t^{(2)} + \int_0^T \eta_t^{(2)} dB_t^{(1)}]\end{aligned}$$

Furthermore, an algebraic calculation tells:

$$\begin{aligned}\langle B\rangle_t &= (B_t^{(1)} + iB_t^{(2)})^2 - 2\int_0^t (B_s^{(1)} + iB_s^{(2)}) d(B_s^{(1)} + iB_s^{(2)}) \\ &= (B_t^{(1)})^2 - (B_t^{(2)})^2 + 2iB_t^{(1)}B_t^{(2)} \\ &- 2[\int_0^t B_s^{(1)} dB_s^{(1)} - \int_0^t B_s^{(2)} dB_s^{(2)} + i\int_0^t B_s^{(1)} dB_s^{(2)} + i\int_0^t B_s^{(2)} dB_s^{(1)}] \\ &= [(B_t^{(1)})^2 - 2\int_0^t B_s^{(1)} dB_s^{(1)}] - [(B_t^{(2)})^2 - 2\int_0^t B_s^{(2)} dB_s^{(2)}] \\ &+ 2i[B_t^{(1)}B_t^{(2)} - \int_0^t B_s^{(1)} dB_s^{(2)} - \int_0^t B_s^{(2)} dB_s^{(1)}] \\ &= \langle B^{(1)}\rangle_t - \langle B^{(2)}\rangle_t + 2i\langle B^{(1)}, B^{(2)}\rangle_t\end{aligned}$$

Here are some basic properties of $\langle B\rangle$. For simplicity, we denote

$$\begin{aligned}\overrightarrow{B_t} &= (B_t^{(1)}, B_t^{(2)}) \\ \overrightarrow{\langle B\rangle}_t &= \begin{pmatrix} \langle B^{(1)}\rangle_t & \langle B^{(1)}, B^{(2)}\rangle_t \\ \langle B^{(1)}, B^{(2)}\rangle_t & \langle B^{(2)}\rangle_t \end{pmatrix}\end{aligned}$$

Remark 5.13. *Since $\overrightarrow{B_t}$ is a two dimensional G-Brownian Motion, $\overrightarrow{\langle B\rangle}_t$ is maximal distributed, so are $Re\langle B\rangle_t$ and $Im\langle B\rangle_t$. Then we can similarly define maximal distribution under complex framework in a trivial way. Then, of course, $\langle B\rangle_t$ is maximal distributed.*

Lemma 5.14. *For $s,t \geq 0$, $\langle B\rangle_{t+s} - \langle B\rangle_t$ is identically distributed with $\langle B\rangle_s$ and independent from Ω_s.*

Proof. Notice $\langle B\rangle_t = \langle B^{(1)}\rangle_t - \langle B^{(2)}\rangle_t + 2i\langle B^{(1)}, B^{(2)}\rangle_t$, and by the definition of independence we have the conclusion. $\square$

Then we need to define integral with respect to $\langle B\rangle_t$. Firstly, we define the map J from $M_G^{1,0}(0,T)_{\mathbb{C}}$ to $L_G^1(\Omega_T)_{\mathbb{C}}$:

$$J(\eta) = \int_0^T \eta_t d\langle B\rangle_t := \sum_{j=0}^{N-1} \xi_j(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}),$$

where $\eta_t(\omega) = \Sigma_{k=0}^{N-1}\xi_k(\omega)I_{[t_k,t_k+1)}(t)$ and then by the following lemma, we could have $\int_0^T \eta_t d\langle B\rangle_t$ for $\eta_t \in M_G^1(0,T)_{\mathbb{C}}$.

Lemma 5.15. *For $\eta_t = \Sigma_{k=0}^{N-1}\xi_k I_{[t_k,t_k+1)}(t) \in M_G^1(0,T)_{\mathbb{C}}$,*

$$\hat{\mathbb{E}}[|J(\eta)|] \leq 4(\bar{\sigma}_1^2 + \bar{\sigma}_3^2 + 2\bar{\sigma}_2^2)\hat{\mathbb{E}}[\int_0^T |\eta_t|dt],$$

where $\bar{\sigma}_i^2 = \sup_{\Lambda\in\Sigma}|\sigma_i^2|, i = 1,2,3,$ and $G(A) = \frac{1}{2}\sup_{\Lambda\in\Sigma}(A,\Lambda)$, with Σ a bounded closed convex subset of 2×2 symmetric matrix and $\Lambda = \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ \sigma_2^2 & \sigma_3^2 \end{pmatrix}$

Proof. We denote

$$\begin{aligned}
\triangle t_j &= t_{j+1} - t_j \\
\triangle\langle B\rangle_j &= \langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}, \\
\triangle\langle B^{(i)}\rangle_j &= \langle B^{(i)}\rangle_{t_{j+1}} - \langle B^{(i)}\rangle_{t_j}, i = 1,2, \\
\triangle\langle B^{(1)}, B^{(2)}\rangle_j &= \langle B^{(1)}, B^{(2)}\rangle_{t_{j+1}} - \langle B^{(1)}, B^{(2)}\rangle_{t_j}.
\end{aligned}$$

We have $\hat{\mathbb{E}}[|\xi_j\triangle\langle B\rangle_j|] \leq \hat{\mathbb{E}}[|Re(\xi_j\triangle\langle B\rangle_j)|] + \hat{\mathbb{E}}[|Im(\xi_j\triangle\langle B\rangle_j)|]$ and then

$$\begin{aligned}
& \hat{\mathbb{E}}[|Re(\xi_j\triangle\langle B\rangle_j)|] \\
= \ & \hat{\mathbb{E}}[|\xi_j^{(1)}\triangle\langle B^{(1)}\rangle_j - \xi_j^{(1)}\triangle\langle B^{(2)}\rangle_j - 2\xi_j^{(2)}\triangle\langle B^{(1)}, B^{(2)}\rangle_j|] \\
\leq \ & (\bar{\sigma}_1^2 + \bar{\sigma}_3^2 + 2\bar{\sigma}_2^2)\hat{\mathbb{E}}[\triangle t_j(|\xi_j^{(1)}| + |\xi_j^{(2)}|)].
\end{aligned}$$

The similar result holds for $Im(\xi_j\triangle\langle B\rangle_j)$, so we have

$$\hat{\mathbb{E}}[|\sum_{j=0}^{N-1}\xi_j\triangle\langle B\rangle_j|] \leq 2(\bar{\sigma}_1^2 + \bar{\sigma}_3^2 + 2\bar{\sigma}_2^2)\hat{\mathbb{E}}[\triangle t_j(|\xi_j^{(1)}| + |\xi_j^{(2)}|)].$$

The inequality $|a| + |b| \leq 2\sqrt{a^2+b^2}$ would finish the proof. $\square$

Proposition 5.16. *Here are some properties for the integral with respect to $\langle B\rangle$.*
(i) For $0\le s\le t$, $\xi\in L_G^2(\Omega_s)_{\mathbb{C}}, Z\in L_G^1(\Omega_s)_{\mathbb{C}}$, we have

$$\hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi(B_t^2-B_s^2)]=\hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi(\langle B\rangle_t-\langle B\rangle_s)]=\hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi(B_t-B_s)^2]$$

(ii) For $\eta\in M_G^{2,0}(0,T)_{\mathbb{C}}$, we have

$$\hat{\mathbb{E}}_{\mathbb{C}}[(\int_0^T\eta_t dB_t)^2]=\hat{\mathbb{E}}_{\mathbb{C}}[\int_0^T\eta_t^2 d\langle B\rangle_t]$$

Proof. For (i), by the definition of $\langle B\rangle_t$ and proposition (3.7), we have

$$\begin{aligned}\hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi(\langle B\rangle_t-\langle B\rangle_s)] &= \hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi(B_t^2-B_s^2+\int_s^t B_r dB_r)]\\ &= \hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi(B_t^2-B_s^2)]\\ &= \hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi((B_t-B_s)^2+2B_s(B_t-B_s))]\\ &= \hat{\mathbb{E}}_{\mathbb{C}}[Z+\xi(B_t-B_s)^2]\end{aligned}$$

For (ii), notice $\hat{\mathbb{E}}_{\mathbb{C}}[Z+2\xi_j(B_{t_{j+1}}-B_{t_j})\xi_i(B_{t_{i+1}}-B_{t_i})]=\hat{\mathbb{E}}_{\mathbb{C}}[Z]$ and then

$$\hat{\mathbb{E}}_{\mathbb{C}}[(\int_0^T\eta_s dB_s)^2]=\hat{\mathbb{E}}_{\mathbb{C}}[(\sum_{j=0}^{N-1}\xi_j(B_{t_{j+1}}-B_{t_j}))^2].$$

By (i), we would have the conclusion.

□

6 Complex *Itô* Formula and Conformal G-Brownian Motion

Now we turn to *Itô* formula under this framework. Firstly we need to set some basic definitions.

Definition 6.1. *For $f=u+iv$ differentiable with real part u and imaginary part v, we define operator ∂f and $\bar{\partial}f$ as:*

$$\partial f:=\frac{\partial}{\partial z}f:=\frac{1}{2}(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y})=\frac{1}{2}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+i(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}))$$

$$\bar{\partial}f:=\frac{\partial}{\partial \bar{z}}f:=\frac{1}{2}(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y})=\frac{1}{2}(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}))$$

Remark 6.2. *For $\hat{B}_i=\alpha_i B_t^{(1)}+i\beta_i B_t^{(2)}$, $i=1,2$, where $\alpha_i,\beta_i\in\mathbb{R}$ and $B_t^{(1)}+iB_t^{(2)}$ is a complex G-Brownian Motion, we have $(\alpha_i B_t^{(1)},\beta_i B_t^{(2)})$ is also*

a two-dimensional $\tilde{G}_i$-Brownian Motion, so $\hat{B}_i$ is also a complex $\tilde{G}_i$-Brownian Motion. Then we can define

$$\langle \hat{B}_1, \hat{B}_2 \rangle_t := \frac{1}{4}(\langle \hat{B}_1 + \hat{B}_2 \rangle_t - \langle \hat{B}_1 - \hat{B}_2 \rangle_t).$$

and we have

$$\begin{aligned}
& \langle \hat{B}_1, \hat{B}_2 \rangle_t \\
= \; & \langle \alpha_1 B^{(1)} + i\beta_1 B^{(2)}, \alpha_2 B^{(1)} + i\beta_2 B^{(2)} \rangle_t \\
= \; & \frac{1}{4}[\langle (\alpha_1 + \alpha_2) B^{(1)} + i(\beta_1 + \beta_2) B^{(2)} \rangle_t - \langle (\alpha_1 - \alpha_2) B^{(1)} + i(\beta_1 - \beta_2) B^{(2)} \rangle_t] \\
= \; & \frac{1}{4}[\langle (\alpha_1 + \alpha_2) B^{(1)} \rangle_t - \langle (\beta_1 + \beta_2) B^{(2)} \rangle_t + 2i \langle (\alpha_1 + \alpha_2) B^{(1)}, (\beta_1 + \beta_2) B^{(2)} \rangle_t \\
- \; & \langle (\alpha_1 - \alpha_2) B^{(1)} \rangle_t + \langle (\beta_1 - \beta_2) B^{(2)} \rangle_t - 2i \langle (\alpha_1 - \alpha_2) B^{(1)}, (\beta_1 - \beta_2) B^{(2)} \rangle_t] \\
= \; & \alpha_1 \alpha_2 \langle B^{(1)} \rangle_t - \beta_1 \beta_2 \langle B^{(2)} \rangle_t + i(\alpha_1 \beta_2 + \beta_1 \alpha_2) \langle B^{(1)}, B^{(2)} \rangle_t.
\end{aligned}$$

In particular, $\langle B, \bar{B} \rangle_t = \langle B^{(1)} \rangle_t + \langle B^{(2)} \rangle_t$.

Furthermore, $\int_0^T \eta_s \langle B_1, B_2 \rangle_s$ also explains itself.

Theorem 6.3. *(Complex Version of Itô's Lemma) For*

$$Z_t = Z_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B \rangle_s + \int_0^t \beta_s dB_s,$$

where α, β, η are all bounded processes in $M_G^2(0,T)_{\mathbb{C}}$, and any function f, which is twice continuously differentiable and satisfies polynomial growth condition for the second order derivatives, we have the following equation in $L_G^2(\Omega_t)_{\mathbb{C}}$, $\forall t \geq 0$:

$$\begin{aligned}
f(Z_t) - f(Z_0) \; = \; & \int_0^t \partial f(Z_s) dZ_s + \int_0^t \bar{\partial} f(Z_s) d\bar{Z}_s + \int_0^t \partial\partial f(Z_s) d\langle Z \rangle_s \\
+ \; & \int_0^t \bar{\partial}\bar{\partial} f(Z_s) d\langle \bar{Z} \rangle_s + 2\int_0^t \partial\bar{\partial} f(Z_s) d\langle Z, \bar{Z} \rangle_s \\
= \; & \int_0^t (\partial f \alpha_s + \bar{\partial} f \bar{\alpha}_s) ds + \int_0^t \partial f \beta_s dB_s + \int_0^t \bar{\partial} f \bar{\beta}_s d\bar{B}_s + \int_0^t \partial f \eta_s d\langle B \rangle_s + \int_0^t \bar{\partial} f \bar{\eta}_s d\overline{\langle B \rangle}_s \\
+ \; & \int_0^t \partial\partial f \beta_s^2 d\langle B \rangle_s + \int_0^t \bar{\partial}\bar{\partial} f \bar{\beta}_s^2 d\langle \bar{B} \rangle_s + 2\int_0^t \partial\bar{\partial} f |\beta_s|^2 d\langle B, \bar{B} \rangle_s
\end{aligned}$$

Proof. Notice $\overline{\langle B \rangle}_s = \langle B^{(1)} \rangle_s - \langle B^{(2)} \rangle_s - 2i \langle B^{(1)}, B^{(2)} \rangle_s$. The proof can be done by a review of *itô* formula in the real case and a careful algebraic calculation. □

Remark 6.4. *The first equation in the above theorem is formal to avoid the complex structure of the second equation. See [9], [10].*

Remark 6.5. *If f is twice continuously differentiable, we have $\frac{\partial}{\partial x}\frac{\partial}{\partial y}f = \frac{\partial}{\partial y}\frac{\partial}{\partial x}f$, and further $\partial\bar{\partial}f = \bar{\partial}\partial f = \frac{1}{4}\Delta f$, where Δ is the laplace operator, i.e. $\Delta := (\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2$.*

We know that a complex G-Brownian Motion can be viewed as a two-dimensional real G-Brownian Motion, which includes too many elements to get better properties. Here is a special kind of complex G-Brownian Motion, called conformal G-Brownian Motion, which is the main object of complex stochastic analysis.

Definition 6.6. *A complex G-Brownian Motion is called a conformal G-Brownian Motion if $\langle B\rangle_t = 0$ in $L_G^2(\Omega_t)_{\mathbb{C}}$ for any $t \geq 0$.*

Remark 6.7. *Notice $\langle B\rangle_t = \langle B^{(1)}\rangle_t - \langle B^{(2)}\rangle_t + 2i\langle B^{(1)}, B^{(2)}\rangle_t$, so $\langle B\rangle_t \equiv 0$ if and only if $\langle B^{(1)}\rangle_t = \langle B^{(2)}\rangle_t$ and $\langle B^{(1)}, B^{(2)}\rangle_t = 0$. This means the real part and the imaginary part moves as the same rate (identically distributed) and they are irrelevant. In the classical case, a complex Brownian Motion is surely conformal, since its real part and imaginary part are independent. However, things are a little different under G-framework. We cannot say $B^{(1)}$ and $B^{(2)}$ are independent under G-framework.*

Example 6.8. *For a random vector $X = (X_1, X_2)$, where X_1 is a real G-normal distributed variable with $\bar{\sigma}^2 > \underline{\sigma}^2$, and X_2 is an independent copy of X_1, we can claim that X fails to be a real G-normal distributed vector, which can be easily checked by the definition of real G-normal distribution. In fact, if $\bar{X} = (\bar{X}_1, \bar{X}_2)$ is an independent copy of X, $\bar{X}_2$ is independent of $\bar{X}_1$ by the definition of independence, so for $\varphi(x, y) = x^2 y$, we have*

$$\begin{aligned}
\hat{\mathbb{E}}[\varphi(X + \bar{X})] &= \hat{\mathbb{E}}[(X_1 + \bar{X}_1)^2(X_2 + \bar{X}_2)] \\
&= \hat{\mathbb{E}}[X_2^+\bar{\sigma}^2 - X_2^-\underline{\sigma}^2] \\
&= (\bar{\sigma}^2 - \underline{\sigma}^2)\hat{\mathbb{E}}[X_2^+] \\
&= \frac{1}{2}(\bar{\sigma}^2 - \underline{\sigma}^2)\hat{\mathbb{E}}[|X_2|] > 0
\end{aligned}$$

while $\hat{\mathbb{E}}[\varphi(\sqrt{2}X)] = 0$.
This means a nontrivial two dimensional G-normal distributed vector fails to have independent elements, so does a complex G-normal distributed variable.

Before giving a description of complex G-conformal Brownian Motion, we need a simple fact.

Lemma 6.9. *If X is a real maximal distributed n-dimensional vector and satisfies*

$$\hat{\mathbb{E}}[\varphi(X)] = \varphi(0)$$

for any $\varphi \in C_{l.lip}(\mathbb{R}^n)$. Then we have $X = 0$, q.s

Proof. Take $\varphi(x) = |x|^2$. Since X is maximal distributed,

$$\hat{\mathbb{E}}[\varphi(X)] = \sup_{(X_1,\cdots,X_n)\in V}(X_1^2 + \cdots + X_n^2) = 0$$

where V is a convex closed subset of $\mathbb{R}^n$ (see remark(2.5)). It must be $V = \{0\}$, and we have the conclusion. $\square$

Here is a description of conformal G-Brownian Motion.

Theorem 6.10. *A complex G-Brownian Motion B_t is conformal if and only if $G(\cdot)$ has the following expression :*

$$G(A) := \hat{\mathbb{E}}[\overrightarrow{B_t}A\overrightarrow{B_t}^T] = \frac{1}{2}\sup_{\Lambda\in\Sigma}(A,\Lambda)$$

where $\Sigma = \{\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} |\sigma^2 \in [\underline{\sigma}^2, \bar{\sigma}^2]\}$*, and* $\bar{\sigma}^2 \geq \underline{\sigma}^2 \geq 0$.

Proof. According to remark (6.7), the only if part is simple. In fact, if we denote

$$\Lambda = \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ \sigma_2^2 & \sigma_3^2 \end{pmatrix},$$

since $\langle B^{(1)}\rangle_t = \langle B^{(2)}\rangle_t$ and $\langle B^{(1)}, B^{(2)}\rangle_t = 0$, we have

$$\hat{\mathbb{E}}[f(\langle B^{(1)}\rangle_t - \langle B^{(2)}\rangle_t)] = \sup_{\Lambda\in\Sigma} f(\sigma_1^2 - \sigma_3^2) = f(0)$$

and

$$\hat{\mathbb{E}}[g(\langle B^{(1)}, B^{(2)}\rangle_t)] = \sup_{\Lambda\in\Sigma} g(\sigma_2^2) = g(0),$$

for any $f, g \in C_{l.lip}(\mathbb{R})$. It follows that Σ must have the above expression.

For the if part, notice B_t is uniquely determined by $G(A)$, and

$$\hat{\mathbb{E}}[\varphi(\overrightarrow{\langle B\rangle}_t)] = \sup_{\Lambda\in\Sigma}\varphi(\Lambda).$$

We have

$$\hat{\mathbb{E}}[\varphi(\langle B^{(1)}\rangle_t - \langle B^{(2)}\rangle_t)] = \sup_{\Lambda\in\Sigma}\varphi(\sigma^2 - \sigma^2) = \varphi(0)$$

and

$$\hat{\mathbb{E}}[\varphi(\langle B^{(1)}, B^{(2)}\rangle_t)] = \sup_{\Lambda\in\Sigma}\varphi(0) = \varphi(0),$$

for any $\varphi \in C_{l.lip}(\mathbb{R})$. By lemma (6.9), we get the conclusion. $\square$

We give the definition of G-martingale under complex case in a trivial way.

Definition 6.11. *For a complex process $(M_t)_{t\geq 0}$, it is called a complex G-martingale if $M_t \in L^1_G(\Omega_t)_{\mathbb{C}}$, and*

$$\hat{\mathbb{E}}_{\mathbb{C}}[M_t|\Omega_s] = M_s,$$

for $0 \leq s \leq t$.

Here is a property of analytic function, which we will use in the next.

Lemma 6.12. *f is continuously differentiable complex function. Then f is analytic if and only if $\partial f = 0$. In this case,*

$$f'(z) = \partial f(z),$$

here $f'(z)$ means the derivative of f in the complex sense.

Proof. Notice that $\partial f = 0$ if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, when $f = u + iv$. The conclusion follows from Cauchy-Riemann equation.

□

Corollary 6.13. *Suppose B_t is a conformal G-Brownian Motion, and f is analytic. Then $f(B_t)$ is a symmetric complex G-martingale.*

Proof. In this case, by $it\hat{o}'s$ formula and the above lemma, we have

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s.$$

□

Example 6.14. *Suppose B_t is conformal. Then B_t^2 is a martingale. In face, $B_t^2 = (B_t^{(1)})^2 - (B_t^{(2)})^2 + 2iB_t^{(1)}B_t^{(2)}$, so we have*

$$\begin{aligned}
\hat{\mathbb{E}}_{\mathbb{C}}[B_t^2|\Omega_s] &= \hat{\mathbb{E}}[(B_t^{(1)})^2 - (B_t^{(2)})^2|\Omega_s] + 2i\hat{\mathbb{E}}[B_t^{(1)}B_t^{(2)}|\Omega_s] \\
&= \hat{\mathbb{E}}[(B_t^{(1)})^2 - \langle B^{(1)}\rangle_t - ((B_t^{(2)})^2 - \langle B^{(2)}\rangle_t) + \langle B^{(1)}\rangle_t - \langle B^{(2)}\rangle_t|\Omega_s] \\
&+ 2iB_s^{(1)}B_s^{(2)} \\
&= (B_s^{(1)})^2 - (B_s^{(2)})^2 + 2iB_s^{(1)}B_s^{(2)} \\
&= B_s^2
\end{aligned}$$

In fact, we can take this conclusion further to get the conformal invariance by considering martingale with the form $M_t = \int_0^t \eta_u dB_u$, where $\eta_u \in M_G^2(0,T)_{\mathbb{C}}$. Since M_t is a symmetric martingale, we can define the quadratic variation in the old fashion way: the limit point under norm of $L_G^2(\Omega_T)_{\mathbb{C}}$, and then we would have

$$\langle M\rangle_t = \int_0^t \eta_s^2 d\langle B\rangle_s.$$

Definition 6.15. *A complex martingale M_t is conformal if $\langle M\rangle_t = 0$.*

Obviously, a conformal G-Brownian motion is a conformal martingale. Furthermore, we suppose $\eta_u \in M_G^4(0,T)_{\mathbb{C}}$. If B_t is conformal, by $it\hat{o}'$s lemma and B-D-G inequity, we have $M_t^2 = 2\int_0^2 M_s \eta_s dB_s$, so $\langle M^2 \rangle_t = 4\int_0^t M_s^2 \eta_s^2 d\langle B\rangle_s$. Notice $d\langle M\rangle_s = \eta_s^2 d\langle B\rangle_s$, and we get $\langle M^2\rangle_t = 4\int_0^t M_s^2 d\langle M\rangle_s$. In conclusion, we get:

Corollary 6.16. *B_t is conformal and M_t is defined as above. Then M_t, M_t^2 are conformal martingales.*

Corollary 6.17. *If $M_t = \int_0^t \eta_s dB_s$ is conformal, with values in an open set E, and f is a bounded analytic function with bounded first order derivative on E, polynomial growth of the second order derivative, then $f(M_t)$ is conformal. Furthermore,*

$$\langle f(M), \overline{f(M)}\rangle_t = \int_0^t f'(M_s)\overline{f'(M_s)} d\langle M, \bar{M}\rangle_s$$

Proof. By it$\hat{o}'s$ lemma, $f(M_t) = \int_0^t f'(M_s)\eta_s dB_s$, so

$$\begin{aligned}\langle f(M)\rangle_t &= \int_0^t (f'(M_s))^2 \eta_s^2 d\langle B\rangle_s \\ &= \int_0^t (f'(M_s))^2 d\langle M\rangle_s \\ &= 0\end{aligned}$$

The equation follows from the fact that $f(M)$ is also a symmetric martingale. □

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